Closing tonight: 14.3(1)
Closing Tues: 14.3(2), 14.4
Closing Thurs: 14.7(1)
Office Hours: 12:30-2:00pm, MSC

14.3/14.4 Partial Der. & Tangent Planes

Note: A variable can be treated as

1. A constant (constant term or coef)
2. An independent variable (input)
3. A dependent variable (output)

Entry Task: Find the derivatives

a) $y = f(x) = x^2 e^x$

$$\frac{dy}{dx} =$$

b) An object's motion (x,y) = (x(t),y(t)) satisfies y = x^2 for all times.

$$\frac{dy}{dt} =$$

c) $z = x^2 + y^3 e^{6y} - 5xy^4 + \ln(w)$

$$\frac{\partial z}{\partial x} =$$

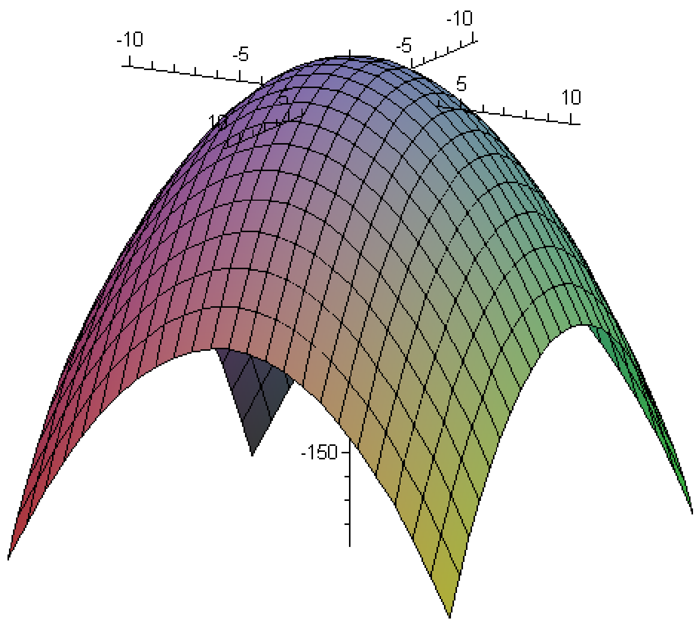
$$\frac{\partial z}{\partial y} =$$

$$\frac{\partial z}{\partial w} =$$

d) $x^2 + y^3 = 1, \frac{dy}{dx} = ??$

e) $x^2 + t^3 + y^2 - z^2 = 1, \frac{\partial z}{\partial x} = ??$

Graphical Interpretation of Partial Der: Pretend you are skiing on the surface $z = f(x, y) = 15 - x^2 - y^2$.



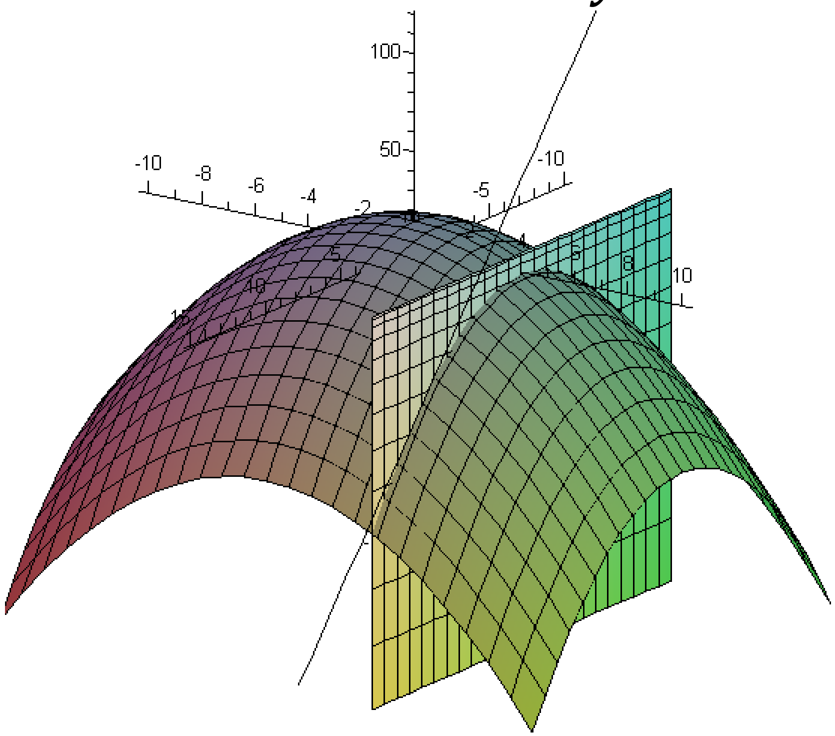
Exercise:

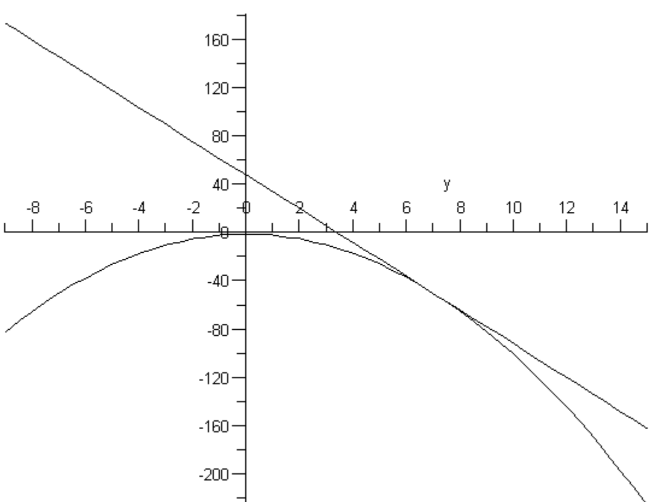
1. Find $f_x(x, y)$ and $f_y(x, y)$

2. Assume you are standing on the point on the surface corresponding to (x,y) = (7,4). Compute:
   i) $f(7,4) =$
   ii) $f_x(7,4) =$
   iii) $f_y(7,4) =$

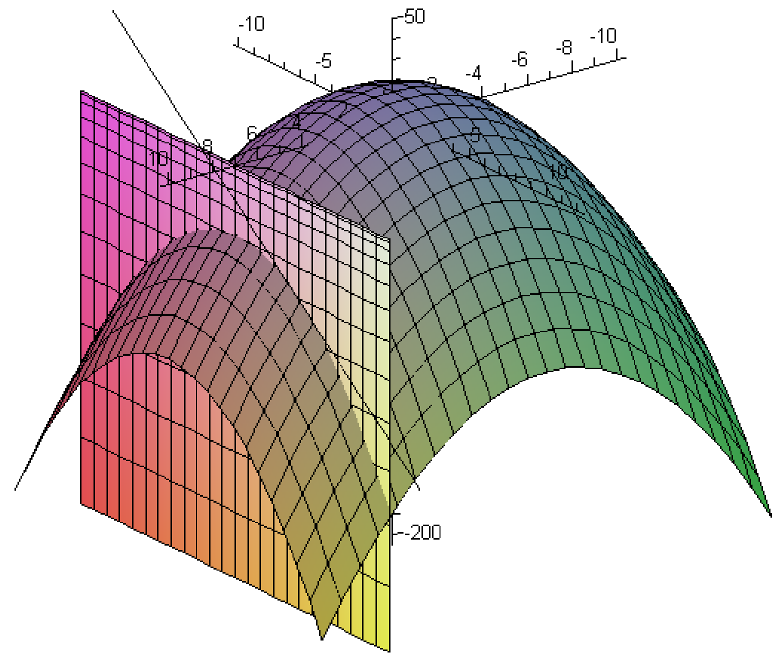
   What do these three numbers represent?

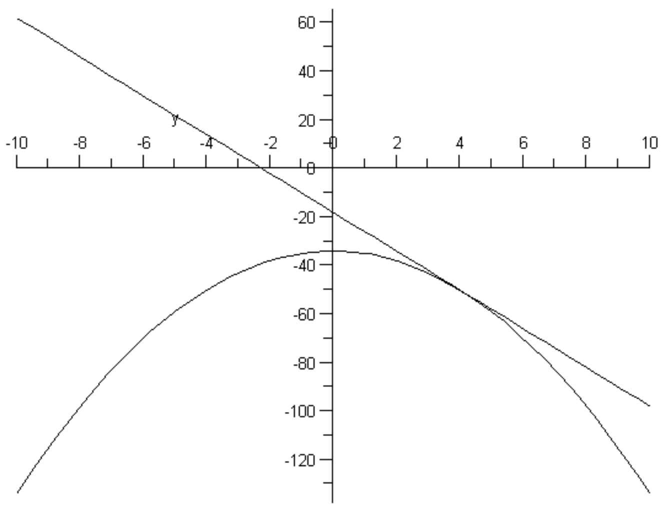
The plane y = 4 intersecting the surface $z = 15 - x^2 - y^2$.





The plane x = 7 intersecting the surface $z = 15 - x^2 - y^2$.





Second Partial Derivatives

Concavity in *x*-direction:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = f_{xx}(x, y)$$

Concavity in *y*-direction:

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = f_{yy}(x, y)$$

Mixed Partials:

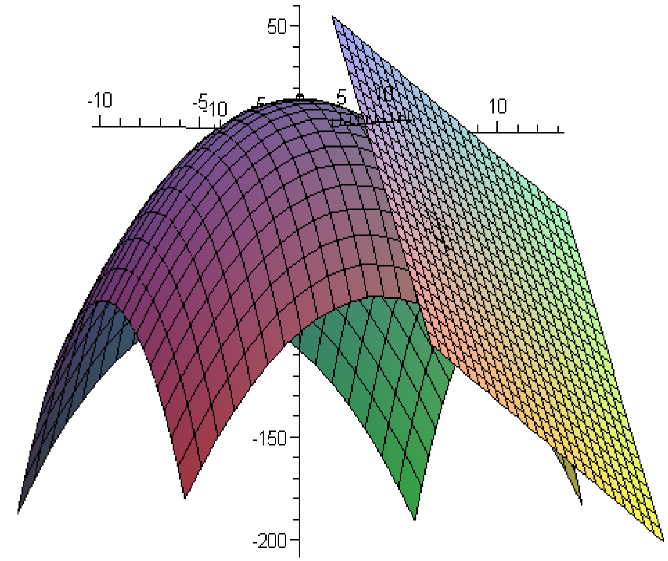
$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = f_{xy}(x, y)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = f_{yx}(x, y)$$

Example: Find all second partials for

$z = f(x, y) = x^4 + 3x^2y^3 + y^5$

14.4 Tangent Planes (linear approx.)



The <u>tangent plane</u> to a surface at a point is the plane that contains all tangent lines at that point. It is given by:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example: Find the tangent plane to

$z = f(x,y) = 15 - x^2 - y^2$ at (7,4)

Recall:

$f(7,4) =$

$f_x(7,4) =$

$f_y(7,4) =$

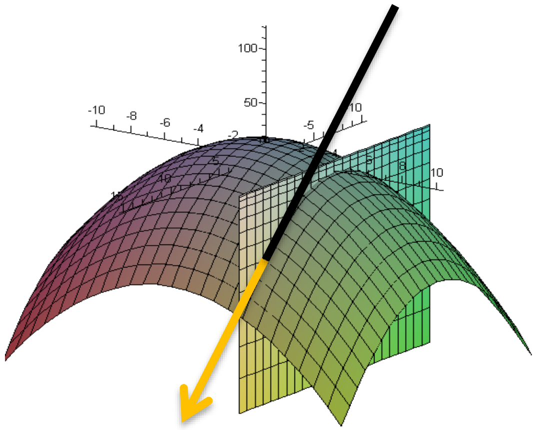
Derivation of Tangent Plane

The plane goes thru (7, 4, -50).
Now we need a normal vector.

Note:

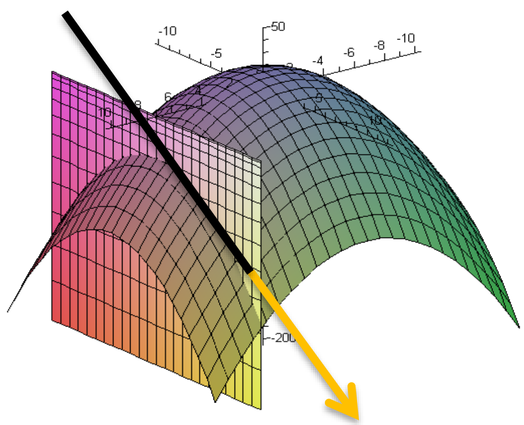
$f_x(x,y) = -2x$

$f_x(7,4) = -14$



$f_y(x,y) = -2y$

$f_y(7,4) = -8$



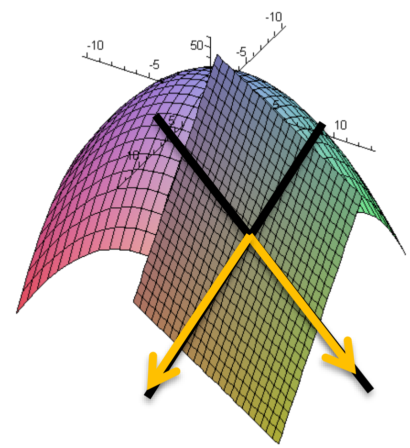
Thus, we can get two vectors that are parallel to the plane:

$$<1,0,f_x(x_0,y_0)> = <1,0,-14>$$

$$<0,1,f_y(x_0,y_0)> = <0,1,-8>$$

So a normal vector is given by

$<1,0,-14> \times <0,1,-8> = <14, 8, 1>$



Tangent Plane:

$$14(x-7) + 8(y-4) + (z+50) = 0$$

Which we rewrite as:

$$z + 50 = -14(x-7) - 8(y-4)$$

General Derivation

In general, for $z = f(x,y)$ at (x_0, y_0) by:

1. $z_0 = f(x_0, y_0)$ = height.

2. $\langle 1,0, f_x(x_0, y_0)\rangle =$ 'a tangent in *x*-dir.'
   $\langle 0,1, f_y(x_0, y_0)\rangle =$ 'a tangent in *y*-dir.'

3. Normal to surface:
   $$\langle 1,0, f_x(x_0, y_0)\rangle \times \langle 0,1, f_y(x_0, y_0)\rangle$$
   $$= \langle -f_x(x_0, y_0), -f_y(x_0, y_0), 1\rangle$$

Tangent Plane:

$$-f_x(x_0, y_0)(x - x_0) - f_y(x_0, y_0)(y - y_0) + (z - z_0) = 0$$

which we typically write as:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example: Find the tangent plane for

$$f(x,y) = x^2 + 3y^2x - y^3$$

at $(x,y) = (2,1)$.

An Application of the Tangent Planes

Linear Approximation

"Near" the point (x_0,y_0) the tangent plane and surface z-values are close.

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0),$$

which is the same as

$$L(x, y) = z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Idea:

$$f(x, y) \approx L(x, y) \text{ for } (x, y) \approx (x_0, y_0)$$

Example:

Use the linear approximation to $f(x,y) = x^2 + 3y^2x - y^3$ at $(x,y) = (2,1)$ to estimate the value of $f(1.9, 1.05)$.